

# EMBEDDING ALMOST-COMPLEX MANIFOLDS IN ALMOST-COMPLEX EUCLIDEAN SPACES

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**ABSTRACT.** We show that any compact almost-complex manifold  $(M, J)$  of complex dimension  $m$  can be pseudo-holomorphically embedded in  $\mathbb{R}^{6m}$  equipped with a suitable almost-complex structure  $\tilde{J}$ .

**KEYWORDS:** embedding, almost-complex structure, manifold, pseudo-holomorphic embedding.

**AMS CLASSIFICATION:** 32Q60, 32H02.

## 1. INTRODUCTION

An almost-complex structure on a  $2n$ -dimensional smooth manifold  $M$  is a tensor  $J \in \text{End}(TM)$  such that  $J^2 = -\text{id}$ . If  $M$  is oriented we say that  $J$  is *positive* if the orientation induced by  $J$  on  $M$  agrees with the given one. An almost-complex structure is called *integrable* if it is induced by a holomorphic atlas. In dimension two any almost-complex structure is integrable, while in higher dimension this is far from true. A smooth map  $f: N \rightarrow M$  between two almost-complex manifolds  $(N, J')$ ,  $(M, J)$  is called *pseudo-holomorphic* if  $J \circ Tf = Tf \circ J'$ , where  $Tf: TN \rightarrow TM$  is the tangent map of  $f$ . When the map  $f$  is an embedding,  $(N, J')$  is said to be an *almost-complex submanifold* of  $(M, J)$ . In this case we can identify  $N$  with its image  $f(N) \subset M$  and the almost-complex structure  $J'$  with the restriction of  $J$  to  $TN \cong T(f(N)) \subset TM$ .

If we equip  $\mathbb{R}^{2n}$  with the canonical complex structure, that is to say  $\mathbb{R}^{2n} \cong \mathbb{C}^n$ , then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds pseudo-holomorphically embedded in  $\mathbb{R}^{2n}$  equipped with an integrable or non-integrable almost-complex structure.

In [2] Calabi and Eckmann constructed the first examples of compact, simply connected complex manifolds  $M_{p,q}$  which are not algebraic. Topologically  $M_{p,q}$  is the product  $S^{2p+1} \times S^{2q+1}$ . Then by deleting a point on each factor one obtains a complex structure  $J$  on  $\mathbb{R}^{2p+2q+2}$ . In section 5 of [2] it was shown that when  $p, q > 1$  there exists a complex torus as a complex submanifold of  $(\mathbb{R}^{2p+2q+2}, J)$  [2, p. 499]. It follows that the Calabi-Eckmann complex structure  $J$  on  $\mathbb{R}^{2n}$  cannot be tamed by any symplectic form and in particular cannot be Kähler. Calabi and Eckmann also observed that the only holomorphic functions on  $(\mathbb{R}^{2p+2q+2}, J)$  are the constants answering negatively to a question raised by Bochner about the uniformization of complex structures on  $\mathbb{R}^{2n}$ . In [1] Bryant constructed pseudo-holomorphic non-constant maps  $\varphi: M^2 \rightarrow S^6$  for any compact Riemann surface  $M^2$ , where  $S^6$  is equipped with the almost-complex structure induced by

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the octonion multiplication. These maps realize compact Riemann surfaces as pseudo-holomorphic singular curves in  $S^6$ .

In [3] it was shown that any almost-complex torus  $\mathbb{T}^n = \mathbb{R}^{2n}/\Lambda$  can be pseudo-holomorphically embedded into  $(\mathbb{R}^{4n}, J_\Lambda)$  for a suitable almost-complex structure  $J_\Lambda$ . It follows that any compact Riemann surface can be realized as a pseudo-holomorphic curve of some  $(\mathbb{R}^{2n}, J)$ , where  $J$  is a suitable almost-complex structure.

In this paper we prove the following general theorem.

**Theorem 1.** *Any compact almost-complex manifold  $(M, J)$  of real dimension  $2m$  can be pseudo-holomorphically embedded in  $(\mathbb{R}^{6m}, \tilde{J})$  for a suitable positive almost-complex structure  $\tilde{J}$ .*

In particular, any compact Riemann surface can be realized as a pseudo-holomorphic curve in  $(\mathbb{R}^6, \tilde{J})$ . In [3] was shown that the torus is the only compact Riemann surface that can be pseudo-holomorphically embedded in  $(\mathbb{R}^4, \tilde{J})$  for some  $\tilde{J}$ .

## 2. PRELIMINARIES

The space of positive linear complex structures on  $\mathbb{R}^{2n}$  is diffeomorphic to the homogeneous space  $\tilde{\mathfrak{J}}(n) = GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$  and is homotopy equivalent to  $\mathfrak{J}(n) = SO(2n)/U(n)$ . So, an almost-complex structure  $J$  on  $\mathbb{R}^{2n}$  can be regarded as a smooth map  $J : \mathbb{R}^{2n} \rightarrow \tilde{\mathfrak{J}}(n)$ .

**Lemma 2.** *Let  $M \subset \mathbb{R}^{2n}$  be a closed submanifold and let  $J : M \rightarrow \tilde{\mathfrak{J}}(n)$  be a smooth map. Then there exists a smooth extension  $\tilde{J} : \mathbb{R}^{2n} \rightarrow \tilde{\mathfrak{J}}(n)$  if and only if  $J$  is homotopic to a constant.*

*Proof.* The ‘only if’ part follows immediately from the fact that  $\mathbb{R}^{2n}$  is contractible.

Let us prove the ‘if’ part. Consider a smooth homotopy  $H : M \times [0, 1] \rightarrow \tilde{\mathfrak{J}}(2n)$  such that  $H_0(x) = J_0$  for all  $x \in M$ , and  $H_1 = J$  where  $H_t(x) = H(x, t)$  and  $J_0 \in \tilde{\mathfrak{J}}(n)$ . We can extend  $H$  to  $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n} \times [0, 1]$  by setting  $H(x, 0) = J_0$  for any  $x \in \mathbb{R}^{2n}$ . By the homotopy extension property [4, Chapter 0] there exists  $\tilde{H} : \mathbb{R}^{2n} \times [0, 1] \rightarrow \tilde{\mathfrak{J}}(n)$  which extends  $H$ . We conclude the proof by setting  $\tilde{J} = \tilde{H}_1$ .  $\square$

Let  $(M, J)$  be an almost-complex manifold. The strategy to prove Theorem 1 will be to choose an arbitrary embedding  $f : M \hookrightarrow \mathbb{R}^{6m}$ , which exists for the weak Whitney embedding theorem, and to show that  $J$  extends to the pullback  $f^*(T\mathbb{R}^{6m})$  and this extension is null-homotopic.

Consider the standard filtration  $SO(1) \subset SO(2) \subset \dots$ . Since  $SO(n-1)$  contains the  $(n-2)$ -skeleton of  $SO(n)$  (because the standard fibration  $SO(n) \rightarrow S^{n-1}$ ) it follows that the  $k$ -skeleton of  $SO(n)$  is contained on  $SO(k+1)$  for  $0 \leq k \leq n-2$ .

Since  $SO(n) \subset U(n)$  it follows that  $U(n)$  contains the  $(n-1)$ -skeleton of  $SO(2n)$  for  $n \geq 1$ . Then the homomorphism induced by the inclusion  $i_* : \pi_j(U(n)) \rightarrow \pi_j(SO(2n))$  is an isomorphism for  $j \leq n-2$  and is an epimorphism for  $j = n-1$ .

From the homotopy exact sequence of the fibre bundle  $SO(2n) \rightarrow \mathfrak{J}(n)$  given by the projection map it follows that  $\pi_j(\tilde{\mathfrak{J}}(n)) \cong \pi_j(\mathfrak{J}(n)) \cong 0$  for  $j \leq n-1$ .

**Definition 3.** *A space  $X$  is said to be  $n$ -connected if  $\pi_j(X) \cong 0$  for all  $j \leq n$ .*

In particular, 0-connected means path-connected.

From the above considerations we have that  $\tilde{\mathfrak{J}}(n)$  is  $(n-1)$ -connected. The following proposition is well-known in the theory of CW-complexes.

**Proposition 4.** *If  $X$  is  $n$ -connected then any map  $Y \rightarrow X$  defined on a CW-complex  $Y$  of dimension  $\leq n$  is homotopic to a constant.*

Also the following proposition is standard, and we give only the idea of the proof.

**Proposition 5.** *Let  $\xi : E \rightarrow M$  be an oriented real vector bundle of rank  $2k$  over an  $m$ -manifold  $M$ . If  $k \geq m$  then  $\xi$  admits a positive complex structure.*

*Proof.* Consider the bundle  $\xi^{\mathfrak{J}} : \tilde{\mathfrak{J}}(E) \rightarrow M$  with fibre  $\tilde{\mathfrak{J}}(k)$  induced by  $\xi$ . Namely, for any  $p \in M$  the fibre of  $\xi^{\mathfrak{J}}$  over  $p$  is the space of positive linear complex structures on  $\xi^{-1}(p)$ . Since  $\tilde{\mathfrak{J}}(k)$  is  $(k-1)$ -connected, it follows that  $\xi^{\mathfrak{J}}$  admits a section if  $k \geq m$ , see [7, Part III]. This section is a positive complex structure on  $\xi$ .  $\square$

Let  $f : M \rightarrow \mathbb{R}^N$  be an immersion. The normal bundle  $\nu_f(M)$  is, as usual, the orthogonal complement of  $TM$  in  $f^*(T\mathbb{R}^N)$ , that is to say:

$$f^*(T\mathbb{R}^N) = TM \oplus \nu_f(M).$$

If  $M$  is oriented then the normal bundle can be equipped with a canonical orientation, namely that which makes the splitting of  $f^*(T\mathbb{R}^N)$  into a Whitney sum of oriented fibre bundles, where  $\mathbb{R}^N$  is considered with the standard orientation.

### 3. PROOF OF THE MAIN RESULTS

**Theorem 6.** *Let  $M \subset \mathbb{R}^{2n}$  be a submanifold of even dimension endowed with an almost-complex structure  $J$ . If the normal bundle of  $M$  in  $\mathbb{R}^{2n}$  admits a positive complex structure with respect to the canonical orientation, then for any  $k \geq \max(0, \dim_{\mathbb{R}} M - n + 1)$  there exists an almost-complex structure  $\tilde{J}$  on  $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$  such that  $M \times \{0\} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2k}$  is an almost-complex submanifold.*

*Proof.* Let us choose a positive complex structure on the normal bundle of  $M$ . Then by taking the Whitney sum with the almost-complex structure on  $M$  we get a complex structure on  $(T\mathbb{R}^{2n})|_M$ . So we obtain a smooth map  $J : M \rightarrow \tilde{\mathfrak{J}}(n)$ .

In view of Lemma 2 our target is to get a  $J$  null-homotopic. This is so if  $\dim_{\mathbb{R}} M \leq n-1$  because  $\tilde{\mathfrak{J}}(n)$  is  $(n-1)$ -connected and Proposition 4.

If  $\dim_{\mathbb{R}} M > n-1$  we take the product  $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ , where  $\mathbb{R}^{2k}$  is endowed with the standard complex structure, and we embed  $M$  as  $M \times \{0\}$ . We get a complex structure on the normal bundle of  $M$  in  $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$  in the obvious way. So we obtain a map  $J_k : M \rightarrow \tilde{\mathfrak{J}}(n+k)$ . It follows that  $J_k$  is homotopic to a constant if  $k \geq \dim_{\mathbb{R}} M - n + 1$ . In this case  $J_k$  extends on  $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$  by Lemma 2.  $\square$

It follows that if  $(M, J)$  is contained in  $\mathbb{C}^n$  with a complex normal bundle and if  $n \geq 2 \dim_{\mathbb{C}} M + 1$ , then there is a positive almost-complex structure  $\tilde{J}$  on  $\mathbb{C}^n$  which makes  $(M, J)$  an almost-complex submanifold of  $(\mathbb{C}^n, \tilde{J})$ .

*Proof of Theorem 1.* Let  $f : M \hookrightarrow \mathbb{R}^{6m}$  be any embedding. The normal bundle  $\nu_f(M)$  has rank  $4m$  and is orientable. By Proposition 5 there is a complex structure on the normal bundle and then we conclude by an application of Theorem 6 with  $k = 0$ .  $\square$

In some cases we can construct an embedding in an euclidean space of lower dimension. Recall that an  $s$ -inverse of the tangent bundle  $TM$  is a vector bundle  $\xi$  such that  $TM \oplus \xi$  is a trivial vector bundle. Observe that if  $f : M \rightarrow \mathbb{R}^N$  is an immersion then the normal bundle  $\nu_f(M)$  is a

real s-inverse of the tangent bundle  $TM$ . The converse also holds and is a Theorem of Hirsch [5], and is given as follows.

**Theorem 7.** (*Hirsch [5]*) *Any s-inverse of  $TM$  is the normal bundle of some immersion  $f : M \rightarrow \mathbb{R}^N$ .*

Let  $\xi$  be a complex s-inverse of  $(TM, J)$  of complex rank  $k$ , namely  $TM \oplus \xi$  is trivial as a real vector bundle. Now Hirsch's Theorem 7 implies that there exists an immersion  $f : M \rightarrow \mathbb{R}^{2(m+k)}$  such that  $\xi$  is isomorphic to  $\nu_f(M)$  as real vector bundles. So  $\nu_f(M)$  carries a complex structure.

Up to a product with some  $\mathbb{R}^{2h}$ , we can assume that  $k \geq m + 1$ , and then  $f$  is regularly homotopic, namely homotopic through immersions, to an embedding  $f_1 : M \rightarrow \mathbb{R}^{2(m+k)}$ . It follows that  $\nu_{f_1}(M) \cong \nu_f(M)$  carries a complex structure. Now apply Theorem 6 to get  $\tilde{J}$ .

If the rank of  $\xi$  satisfies  $m + 1 \leq k \leq 2m - 1$  we get a pseudo-holomorphic embedding in an euclidean space of complex dimension  $m + k < 3m$ .

Let  $(S^6, J)$  be the six-dimensional sphere equipped with the standard almost-complex structure  $J$  obtained from the octonion multiplication. Theorem 1 implies that  $(S^6, J)$  can be pseudo-holomorphically embedded in  $(\mathbb{R}^{18}, \tilde{J})$  for a suitable positive almost-complex structure  $\tilde{J}$ . Using the existence of a low-dimensional s-inverse of  $(TS^6, J)$  we have the following result.

**Corollary 8.** *The almost-complex sphere  $(S^6, J)$  can be pseudo-holomorphically embedded in  $(\mathbb{R}^{14}, \tilde{J})$  for a suitable positive almost-complex structure  $\tilde{J}$ .*

*Proof.* Since  $S^6$  is embedded in  $\mathbb{R}^8$  with trivial normal bundle we conclude by an application of Theorem 6 with  $k = 3$ .  $\square$

Notice that  $(S^6, J)$  can not be pseudo-holomorphically embedded in  $(\mathbb{R}^{12}, \tilde{J})$ . In fact, the Euler class of the normal bundle of any embedding of  $S^6$  in  $\mathbb{R}^{12}$  is zero by a theorem of Whitney, see [6, p. 138]. On the other hand, if  $S^6$  is contained pseudo-holomorphically in  $(\mathbb{R}^{12}, \tilde{J})$ , by a straightforward computation with the Chern class, we obtain for the Euler class  $e(\nu(S^6)) = c_3(\nu(S^6)) = -2\lambda \neq 0$ , which is a contradiction, where  $\lambda \in H^6(S^6)$  is the standard generator.

We conclude with a question. Since our construction is essentially homotopy-theoretic, we are unable to control the integrability of the almost-complex structure  $\tilde{J}$  of Theorem 1. So the following question is very natural.

**Question 9.** *Let  $(M, J)$  be an integrable complex manifold. Is there an embedding of  $(M, J)$  into an integrable  $(\mathbb{R}^{2n}, \tilde{J})$ ?*

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